Analytic Explanation of the method of Maxima and Minima *

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1. What is usually treated in the elements on the method of maxima and minima mainly concerns functions of one single variable quantity, so that, having propounded an arbitrary function V, which had somehow been composited from the variable quantity z and constants, one has to investigate those determinations of the variable z which cause the function V to have a maximum or a minimum value. Sometimes even functions of two or more variables z, y, x are considered and the values to assign to each one, for which the function has the maximum or minimum value, are in question. But the method, by which questions of this last kind are resolved, is completely identical to that applied in the first kind; for, if several variables are involved, successively each one is considered as variable and the value is investigated for which a maximum or minimum results: if this operation was done for each variable, all values will be found, by which the value of the propounded function is maximized or minimized.

2. The situation is not any different, if a function of two variables x and y is propounded and the value to be attributed to y is in question that, having put $x = a$, the function obtains a maximum or minimum value; for; if one writes a for x everywhere, the question is obviously reduced to the first kind of questions. But if that function of the variable x and y was not expanded, but

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is determined by integration, the questions are to be referred to a completely different class and require a completely different method of solution. As if, e.g., Z was an arbitrary function of x and y and an integral formula $\int Zdx$ was propounded, the question is conveniently formulated as follows: *To define the relation among the two variables x and y that the value of that formula becomes maximal or minimal for $x = a$.*

3. How much of a connection there is between questions of this kind and those I referred to the first class, will become clear to the attentive reader in a few moments. For, let V be an expanded function of x and y, for which the value of y has to be investigated, that for $x = a$ the value of the function V becomes maximal or minimal; to solve this question, one can immediately put $x = a$, having done which the value of y will be determined in such a way by applying the first method that it does not depend on an indefinite value of x. But having propounded the integral formula $\int Zdx$, it is not possible to attribute a certain value a to x in the differential formula Zdx but just after the integration; and, for the value of $\int Zdx$ to become maximal or minimal then, a certain determined value to be taken for y does not solve the task, but one has to assign a certain relation among x and y; therefore, since, even though one puts $x = a$ after the integration, the value of the integral $\int Zdx$ nevertheless depends on an indefinite relation among x and y and is determined by all intermediate values of y.

4. But such questions about the formula $\int Zdx$ to be maximized or minimized extend a lot further and are not only not restricted to the cases, in which Z is a function of x and y, but one can take an arbitrary expression for Z which is determined by a relation assumed among x and y. Hence, aside from the variables x and y, Z can also involve a relation of their differentials, and not only those of first order but even of arbitrary higher order: if the ratios of differentials are expressed as follows

$$\frac{dy}{dx} = p, \quad \frac{dp}{dx} = q, \quad \frac{dq}{dx} = r \quad \text{etc.},$$

the quantity Z can be considered as arbitrary function of all of them, i.e. x, y, p, q, r etc. Yes, the quantity Z can even involve other integral formulas somehow; hence many classes of questions arise the method of solution must be accommodated to.

5. Problems of this kind have been begun to be treated on the occasion of the now famous Isoperimetric Problem once considered by Jacob Bernoulli with greatest advances for Analysis; and even though that difficult task was solved by this most ingenious man with extraordinary skill, nevertheless the method he applied extends only to cases, in which the quantity Z, aside from x and y, only involves their first differentials or the letter p and had to be derived in each case from geometrical considerations. But after I had though about the generalization of this branch of analysis a long time, I finally obtained a general method by means of which all problems of this kind, in which the quantity Z not only involves differentials of each order but even integral formulas, can be resolved, which method I explained in an unique book in much detail.

6. But even though this method is of such a nature that its application does not require any geometric figures, nevertheless the invention of this method is derived from the contemplation of curved lines, which is why it did not seem sufficiently natural to me then. For, since this question, in which a relation among x and y is in question, that the integral formula $\int Zdx$, having put $x = a$ after the integration, becomes maximal or minimal, can be propounded without mentioning geometry at all, the adequate solution and one derived from first principles seems to have to be free from each geometrical consideration too. I did not hide this desire in my book and the in the art of analysis very proficient Lagrange in his letters to me communicated that he could fulfill my desire and at the same time explained the foundations of his analysis. Since those seem to contain a lot more, I decided to explain them in my way and elaborate on them here.

7. Therefore, in general let us consider the integral formula $\int Zdx$, in which Z is a function somehow composed from x and y, which involves not only the ratio of their differentials of first order but also of higher order and furthermore also contains one or more integral formulas. But for its determination let us assume the integral to be taken in such a way that it vanishes for $x = 0$; but then after the integration let us attribute a certain value to x, i.e. put $x = a$, and let A be the value, which the integral formula receives then. Now the task is to define that relation among x and y, from which by means of these operations one obtains the maximal or minimal value for A. Therefore, this relation among x and y solving the question must be expressed by a certain finite equation or differential equation of some order, as soon as which was found, the problem is to be considered to be solved.

8. Let us put, as it is customary in analysis, that this relation among x and y in question is already known, so that, whatever definite value is assumed for x, hence also y and therefore even the function Z has a determined value. Now imagine all possible values from $x = 0$ to $x = a$ successively substituted for x, which values proceed in infinitely small segments, but then the values of Z corresponding to these values of x are multiplied by dx, and all these products collected together into one sum will constitute the quantity we indicated by the letter A and which must be a maximum or a minimum. This is to be understood in such a way that, if from another relation among x and y to each value of x other values of y correspond and therefore then Z has other values, from them for A, if it was a maximum, a smaller value will result, but if it was a minimum, a larger value, than if the correct relation among x and y would have been used.

9. But if these variations induced to each value of y are imagined to be infinitely small, then by the nature of maxima and minima hence the quantity A must not undergo any variation; and from this idea the determination of maxima and minima is usually derived. Since we arbitrarily attributed infinitely small variations to the values of y, the change which hence results in all values of Zdx and hence in their total sum A must be calculated, which, having put it equal to zero, will yield an equation containing the nature of the maximum or minimum and hence the relation among x and y in question. Therefore, by this operation the method to investigate maxima or minima of this kind is applied, which hence is based on the same principles as the usual method of maxima and minima; therefore, let us consider in more detail, how it can be derived by mere analytical rules, without auxiliary tools from geometry, since I already solved this same task successfully while resorting to geometry.

10. Therefore, since infinitely small variations induced to the values of y must not cause a change in the value of the quantity A and this has to happen, however these variations are taken, as long as they are just infinitely small, it will suffice to imagine a variation of this kind just in one single value of y and to render the change resulting from this for the quantity A vanishing, from which source also my complete method of maxima and minima was derived. But even though if infinitely small variations of this kind are induced to many values of y, yes even if to all of them, nevertheless the nature of maxima and minima demands that the change the quantity A undergoes is put equal to zero, and this has to happen, no matter how those variations, which are all arbitrary, are assumed.

11. But since in my preceding solution I assumed one single value of y to receive an infinitely small variation, while all remaining values would remain the same, this violated the principle of continuity and this was the main reason the whole investigation could not be completed applying mere rules of analysis, but one had to resort to the contemplation of a geometric figure, in which the values of y are represented by ordinates of a curved line, in order to find the variations the ratio of the differentials of each order will undergo. Therefore, to not violate the principle of continuity, which would obstruct the application of mere analytic tools, let us attribute infinitely small variations to each value of y, which are indefinite in such a way that each one is then determined arbitrarily and even all of them but one can be set equal to zero, having done which one necessarily has to arrive at my first solution.

12. But since we now attribute infinitely small but arbitrary variations not only to one value of y but innumerable values, yes even to all of them, there is no doubt that this method extends a lot further than the preceding and leads to the solution of many other problems, for which the first method would be applied only with a lot of difficulty or even fruitlessly. For, if those variations are determined in a certain way, having translated the question into a problem in geometry, problems of this kind can be resolved, in which not among all curved lines but only among those, even though their amount is still infinite, which are contained in a certain species, one has to assign that curve which has to property of a certain maximum or minimum. Indeed, such questions are often detected to be very difficult; but furthermore, hence justly many increments for analysis can be expected.

13. Therefore, since here we attribute infinitely small variations to each value of y, let us consider two states of the formula $\int Zdx$, in the one of which the values of y are those the relation among x and y requires are contained, but in the other the varied values; for the sake of distinction I will call the first state the *principal* state, the other state the *varied* state. Therefore, the nature of maxima and minima demands the difference between these two states to vanish. Therefore, as in the principal state each value of y, while the variable x is assumed to increase by its differential dx, is imagined to gain the increment dy, so while x remains the same and we proceed from the principal to the varied state, let us set that the value of y is increased by the element δy; hence the difference between these two differential expressions dy and δy is to be carefully noted. But while we attribute increments δy to each value of y, hence making the transition to the varied state, they are to be considered as completely undetermined and independent from the values of y.

14. Having constituted all this, one has to investigate, what an increment the function Z receives for each value of x, while it is translated from the principal into the varied state; this increment is solely caused by the variation of y, if in this translation it is increased by δy. Let us indicate this increment by δZ so that the value of Z translated from the principal into the varied stated is $= Z + \delta Z$; and first it is immediately clear, if the function Z would depend only on x and would not involve y, that it will be $\delta Z = 0$; and therefore, the variable x, however it enters the function Z, does no contribute anything to δZ, but its value results solely from the element δy, by which the variable y is imagined to increase. But here, depending on whether Z either involves only the finite quantities x and y or the ratio of there differentials or even integral formulas, different cases are to be examined.

15. Therefore, first let us put that the function Z involves only the finite quantities x and y so that neither the ratio of the differentials nor integral formulas enter it and in order to define its variation δZ one has to write $y + \delta y$ instead of y in Z everywhere, while x remains unchanged, and so the varied value $Z + \delta Z$ will result; if from this the principal value Z is subtracted, the variation δZ will remain. Therefore, it is manifest that this variation is obtained, if the function Z is differentiated in usual manner with respect to the variable y and one just writes δy instead of dy. Hence, if after the usual differentiation with respect to both variables we had

$$dZ = Mdx + Ndy,$$

for the translation from the principal to the varied state it will be

$$\delta Z = N\delta y;$$

therefore, this variation will be found, if in the ordinary differential one writes 0 instead of dx, but δy instead of dy; and this way we have covered the first case most easily.

16. But further let us see, how for this first case, in which Z is a function of x and y only, the maximal or minimal value of the integral formula $\int Zdx$ must be found. Therefore, since for each value of x the function Z increases by the element $N\delta y$ and hence Zdx by $Ndx\delta y$, the sum of all these infinitesimally small elements from $x = 0$ to $x = a$ will give the variation of A; if that variation is put δA, it will be

$$\delta A = \int Ndx\delta y;$$

since this expression must vanish, whatever law the variations δy follow, it is necessary that for each value of x

$$N = 0.$$

Therefore, this equation expresses the relation among x and y in question, from which the formula $\int Zdx$ obtains either a maximum or minimum value; and hence this property does not only hold in the case $x = a$, but also, whatever other value is attributed to x.

17. Secondly, aside from x and y, let the function Z also contain the ratio of the first differentials, or, having put $\frac{dy}{dx} = p$, let Z be a function of the quantities x, y and p, having differentiated which is usual manner let this expression result

$$dZ = Mdx + Ndy + Pdp.$$

Therefore, one has to find the variation of Z, while it is translated from the principal into the varied state, in which translation the quantity x remains the same, but y is increased by the element δy, but let the element, by which the quantity p grows, be δp. But since $p = \frac{dy}{dx}$, if in the principal state we indicate the value of y corresponding to $x + dx$ by y', it will be $p = \frac{y'-y}{dx}$; now in the translation into the varied state let y increase by the element δy and y' by the element $\delta y'$, and it will be

$$\delta p = \frac{\delta y' - \delta y}{dx}.$$

But $\delta y' - \delta y$ expresses the increment of δy while x increases by its differential dx so that

$$\delta y' - \delta y = d\delta y;$$

but then $\delta y' - \delta y$ can be considered as the variation of $y' - y = dy$, while we go over to the varied state, and so it will also be

$$\delta y' - \delta y = \delta dy;$$

hence, finally it is concluded to be

$$d\delta y = \delta dy \quad \text{and hence} \quad \delta p = \frac{d\delta y}{dx} = \frac{\delta dy}{dx}.$$

18. But in like manner, if Z, aside from x and y, also involves the differentials of higher orders, that, having put

$$\frac{dy}{dx} = p, \quad \frac{dp}{dx} = q, \quad \frac{dq}{dx} = r \quad \text{etc.},$$

Z is an arbitrary function of x, y, p, q, r etc. and differentiating in usual manner we have

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

the increments of the quantities q, r etc., while they are translated from the principal state into the varied state, are determined. For, because of $q = \frac{dp}{dx}$, it will be

$$\delta q = \frac{\delta p' - \delta p}{dx} = \frac{d\delta p}{dx} = \frac{\delta dp}{dx} \quad \text{and likewise} \quad \delta r = \frac{d\delta q}{dx} = \frac{\delta dq}{dx} \quad \text{etc.}$$

But from the results explained above

$$d\delta p = \frac{dd\delta y}{dx} = \frac{d\delta dy}{dx} \quad \text{and} \quad \delta dp = \frac{\delta ddy}{dx},$$

so that

$$\delta q = \frac{dd\delta y}{dx^2} = \frac{d\delta dy}{dx^2} = \frac{\delta ddy}{dx^2},$$

in like manner one will see that

$$\delta r = \frac{ddd\delta y}{dx^3} = \frac{dd\delta dy}{dx^3} = \frac{d\delta ddy}{dx^3} = \frac{\delta dddy}{dx^3},$$

the equality of which formulas of different species is to be carefully noted.

19. Therefore, while the function Z goes over from the principal into the varied state, since the quantity x does not gain an increment, but y the increment δy, then the quantity p the increment $\frac{d\delta y}{dx}$, the quantity q the increment $\frac{dd\delta y}{dx^2}$, the quantity r the increment $\frac{ddd\delta y}{dx^3}$ etc., the increment of the function Z corresponding to this translation will be found by ordinary differentiation by putting

$$dx = 0, \quad dy = \delta y, \quad dp = \frac{d\delta y}{dx}, \quad dq = \frac{dd\delta y}{dx^2}, \quad dr = \frac{ddd\delta y}{dx^3} \quad \text{etc.},$$

whence:

$$\delta Z = N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{ddd\delta y}{dx^3} + \text{etc.}$$

Therefore, hence the variation of the function Z can be defined for each value of x; this form will be illustrated even more, if, as

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

it is noted that, because of $\delta x = 0$,

$$\delta Z = N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.},$$

but, because of $p = \frac{dy}{dx}$, $q = \frac{dp}{dx}$, $r = \frac{dq}{dx}$ etc., further

$$\delta p = \frac{\delta dy}{dx} = \frac{d\delta y}{dx}, \quad \delta q = \frac{\delta dp}{dx} = \frac{d\delta p}{dx} = \frac{dd\delta y}{dx^2}, \quad \delta r = \frac{ddd\delta y}{dx^3}.$$

20. Therefore, since in the translation into the varied state the function Z receives the increment δZ, in the formula $\int Zdx$ the increment $\int \delta Z dx$ will result, which will therefore be

$$\int dx \left(N\delta y + P\frac{d\delta y}{dx} + Q\frac{dd\delta y}{dx^2} + R\frac{ddd\delta y}{dx^3} + \text{etc.} \right),$$

if in which one puts $x = a$ after the integration, one will obtain the variation of A or δA, which put equal to zero will induce the maximal or minimal value to the quantity A. But in this integration one does not consider the transition into the varied state, but it has to be extended throughout all elements of x, since it denotes the sum of all variations corresponding to each value of x from $x = 0$ to $x = a$. Therefore, for the ratio of the differentials indicated by δ not to cause a distraction, write w for δy so that w exhibits an infinitely small arbitrary quantity somehow depending on x; and the above increment to be equated to zero will be

$$\int dx \left(Nw + P\frac{dw}{dx} + Q\frac{ddw}{dx^2} + R\frac{d^3w}{dx^3} + \text{etc.} \right).$$

21. It is perspicuous that in these above differentials the element is assumed to be constant; for, since we put

$$\frac{d\delta p}{dx} \quad \text{or} \quad d\frac{\delta p}{dx} = \frac{dd\delta y}{dx^2},$$

because of $\delta p = \frac{d\delta y}{dx}$, dx was obviously put to be constant. Therefore, having observed this, if we integrate each part of the found integral separately, we will have:

$$\begin{aligned}
\int dx \cdot Nw &= \int Nwdx, \\
\int dx \cdot P\frac{dw}{dx} &= \int Pdw = Pw - \int wdP, \\
\int dx \cdot Q\frac{ddw}{dx^2} &= \int \frac{Qddw}{dx} = \frac{Qdw}{dx} - \frac{wdQ}{dx} + \int \frac{wddQ}{dx}, \\
\int dx \cdot R\frac{d^3w}{dx^3} &= \int \frac{Rd^3w}{dx^2} = \frac{Rddw}{dx^2} - \frac{dRdw}{dx^2} + \frac{wddR}{dx^2} - \int \frac{wd^3R}{dx^2}
\end{aligned}$$

etc.

Therefore, hence the variation in question will consist partly of integral terms, partly of absolute terms, and it will be:

$$\int wdx\left(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\text{etc.}\right)+w\left(P-\frac{dQ}{dx}+\frac{ddR}{dx^2}-\text{etc.}\right)$$
$$+\frac{dw}{dx}\left(Q-\frac{dR}{dx}+\text{etc.}\right)+\frac{ddw}{dx^2}\left(R-\text{etc.}\right).$$

22. Let us substitute δy for w again, and the increment of the integral formula $\int Zdx$, while

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds \quad \text{etc.}$$

and

$$p=\frac{dy}{dx},\quad q=\frac{dp}{dx},\quad r=\frac{dq}{dx},\quad s=\frac{dr}{dx}\quad \text{etc.},$$

while it is translated into its varied state, which can be expressed by $\delta\int Zdx$, will look as follows:

$$\begin{aligned}
&\int dx\delta y\left(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4S}{dx^4}-\text{etc.}\right)\\
&+\delta y\left(P-\frac{dQ}{dx}+\frac{ddR}{dx^2}-\frac{d^3S}{dx^3}+\text{etc.}\right)\\
&+\frac{d\delta y}{dx}\left(Q-\frac{dR}{dx}+\frac{ddS}{dx^2}-\text{etc.}\right)\\
&+\frac{dd\delta y}{dx}\left(R-\frac{dS}{dx}+\text{etc.}\right)\\
&+\frac{d^3\delta y}{dx^3}\left(S-\text{etc.}\right)\\
&+\text{etc.},
\end{aligned}$$

in which formulas, if they involve differentials of higher orders, the differential dx is assumed to be constant. But for δy for each value of x has an arbitrary value.

23. Therefore, if for the value $x = a$ the formula $\int Zdx$ must become a maximum or minimum, the increment just found, if in it one puts $x = a$, must be put equal to zero and this in such a way that it always vanishes, no matter how the variations δy are assumed. Hence even, if such a variation is attributed to one single value of y, which corresponds to a value of x smaller than a, the expression just found must become zero. But then no change is induced to the last values of y corresponding to $x = a$; hence, since for $x = a$ the absolute part of the increment

$$\delta y\left(P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right) + \frac{d\delta y}{dx}\left(Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.}\right) + \frac{dd\delta y}{dx^2}\left(R - \frac{dS}{dx} + \text{etc.}\right) + \text{etc.}$$

only depends on the variation of the last values of y, for them it will be

$$\delta y = 0, \quad d\delta y = 0, \quad dd\delta y = 0 \quad \text{etc.},$$

and hence this part vanishes per se. From this it is necessary that only the integral part is rendered equal to zero and hence it has to be:

$$\int dx\delta y\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}\right) = 0.$$

24. But this expression contains the sum of all variations resulting from each variation of y; but since such a change is assumed to happen in one single value, the whole sum is reduced to this one variation while the remaining ones vanish; hence it is necessary that for this case

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0.$$

But since, no matter in which point this variation is assumed to occur, the nature of maxima and minima in like manner requires that annihilation, it is necessary that for all value of x

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0;$$

therefore, this equation contains the indefinite relation among x and y by which one can arrange that the value of to arise hence of the integral formula $\int Zdx$ becomes maximal or minimal for $x = a$, whence it is plain that this relation does not depend on this quantity a.

25. This is already the same equation I gave for the solution of the same problem once in my book on maxima and minima, but now I have derived it from mere analytic principles; this task was solved so successfully since I assumed the variations, by which one gets to the varied state, to be added to each value of y. Further, the reduction of the integral formulas done in paragraph 21 solves the task completely, in which those were resolved into parts in such a way that the one did not involve the integral sign $\int$, but which remained restricted, but the others only involved the variation $w = \delta y$ without its differentials; this way we achieved that, since each arbitrary variation must be rendered equal to zero separately, the integral formula immediately gave the equation

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \text{etc.} = 0,$$

by which the relation among x and y is expressed indefinitely, but the remaining absolute parts of the increment, only extending to the last values of y, went out of the calculation.

26. And these absolute parts were not found without any benefit, but have an extraordinary use, to which my first method only leading to the equation

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \text{etc.} = 0,$$

is less accommodated; therefore, this new method is to be preferred over that one. To explain that use more clearly, first let Z be a function just of x and y not involving their differentials so that $dZ = Mdx + Ndy$, while $P = 0$, $Q = 0$ etc., and it is obvious that in this case the absolute parts vanish per se, and the problem is perfectly solved, when we immediately set $N = 0$. So, if

$$\left(bb - nxy + \frac{y^3}{c}\right) dx$$

must be a maximum or a minimum, because of

$$N = -nx + \frac{3yy}{c},$$

the question is answered by setting $yy = \frac{1}{3}ncx$, and hence nothing remains to be determined.

27. But if Z furthermore involves $p = \frac{dy}{dx}$, that

$$dZ = Mdx + Ndy + Pdp,$$

then, for $\int Zdx$ to become a maximum or minimum, it is obviously necessary that $N - \frac{dP}{dx} = 0$. But since this equation is a differential equation and even one of second order, if the function P involves the quantity $p = \frac{dy}{dx}$, the integration will contain one or two arbitrary constants and therefore the relation among x and y will not be completely determined. Therefore, I observed already in my book that this relation corresponding to the maximum or minimum must additionally be determined in such a way that for $x = a$ the other variable y has a determined value, and if that differential equation $N - \frac{dP}{dx} = 0$ was of second order, there is another condition to be added. Therefore, in these cases one can add still another condition extending to the most outer values of y to the condition of the maximum or minimum.

28. Therefore, further one can ask, since in these cases the relation among x and y is not completely determined and it can be still exhibited in infinitely many ways, which among all those produces the maximum or minimum. We will be able to deduce this from the absolute part of the increment neglected before, which in this case is $P\delta y$; therefore, its value it has for $x = a$ must also vanish. And hence we understand in general, if $\int Zdx$ must be a maximum or a minimum, while

$$dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds \quad \text{etc.},$$

that the equation

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

must be determined further in such a way that for $x = a$ the following equations are satisfied:

$$P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.} = 0, \quad Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.} = 0,$$
$$R - \frac{dS}{dx} + \text{etc.} = 0, \quad S - \text{etc.} = 0 \quad \text{etc.}$$

29. Since all this will become more clear in an example, let a relation among x and y be in question, that for $x = a$ this formula

$$\int \frac{dx\sqrt{1+pp}}{\sqrt{y}}, \quad \text{while} \quad p = \frac{dy}{dx},$$

has the maximal or minimal value. Therefore, since

$$Z = \frac{\sqrt{1+pp}}{\sqrt{y}},$$

it will be

$$M = 0, \quad N = -\frac{\sqrt{1+pp}}{2y\sqrt{y}} \quad \text{and} \quad P = \frac{p}{\sqrt{y(1+pp)}},$$

and so the equation $N - \frac{dP}{dx} = 0$ or $Ndx - dP = 0$ is to be satisfied, which multiplied by p gives $Ndy = pdP$. But, because of $M = 0$,

$$dZ = Ndy + Pdp \quad \text{and hence} \quad dZ = pdP + Pdp,$$

which integrated yields

$$Z = Pp + C \quad \text{or} \quad \frac{\sqrt{1+pp}}{\sqrt{y}} = \frac{pp}{\sqrt{y(1+pp)}} + C,$$

i.e.

$$\frac{1}{\sqrt{y(1+pp)}} = C = \frac{1}{\sqrt{b}}.$$

Hence we further obtain

$$b = y(1+pp) \quad \text{and} \quad p = \sqrt{\frac{b-y}{y}} = \frac{dy}{dx},$$

so that

$$dx = \frac{ydy}{\sqrt{by - yy}},$$

and by integrating

$$x = c - \sqrt{by - yy} + b \arcsin \frac{2\sqrt{by-yy}}{b}.$$

But for a more complete determination it has to be $P = 0$ for $x = a$, i.e. $p = 0$ and $y = b$; hence, having put $x = a$ and $y = b$, the constant c is defined in such a way that $c = a - \pi b$. And if we want that for $x = a$ also $y = 0$, it must be $b = \frac{a}{\pi}$.

30. Before we accommodate this analytical investigation to the cases, in which the function Z also contains integral formulas, let us examine the analysis we used up to this point more diligently and consider the foundations it is based on more accurately. But this analysis concerns the two variables x and y which partly are referred to the state we called principal, partly to the varied state, so that the one of them x refers to both states equally, the other y on the other hand, while translated from the principal into the varied state, gains the increment δy, but while in the same state x is promoted to $x + dx$, it grows by the usual increment given by the differential dy; hence if the variable y is simultaneously promoted to the varied state and to the place corresponding to $x + dx$, its increment will be $dy + \delta y$. But since x is equally referred to each of both states, it will be $\delta x = 0$.

31. If one now has another arbitrary function V referred to the point x in the principal state and it in the same state is promoted to the place $x + dx$, let us express its increment it hence gains in usual manner by dV. But if it, while the value of x remains the same, is translated from the principal state into the varied state, let us indicate its increment by the new sign δV. If now that function V is somehow composited from the quantities x, y, p, q, r etc., but the quantities p, q, r etc. denote quantities of such a kind, whose increments

$$dp, \quad dq, \quad dr \quad \text{etc.} \quad \text{and} \quad \delta p, \quad \delta q, \quad \delta r \quad \text{etc.}$$

can be exhibited, hence in the usual way of differentiating one can even assign both increments of the function V. For, if for the translation from the place x to the place $x + dx$ in the same state from usual differentiation it was

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr \quad \text{etc.},$$

for the translation from the principal state into the varied state, while x remains the same at each place, as we noted, i.e. $\delta x = 0$, it will be

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r \quad \text{etc.}$$

32. Furthermore, if the differentials of both kinds are mixed, from the results above it is already plain that

$$\delta dV = d\delta V.$$

Therefore, if V already is a differential of the form dU, it will be

$$\delta ddU = d\delta dU = dd\delta U \quad \text{since} \quad \delta dU = d\delta U$$

and in general, no matter in which order the two differentiation signs d and δ are constituted, their order can be permuted arbitrarily; so it will be

$$\delta d^3 V = d\delta d^2 V = d^2 \delta dV = d^3 \delta V.$$

But since here we want to consider one single varied state, the transition to which is indicated δ, this sign can never be contained more than once in compositions of this kind; but it is always allowed to move the sign δ to the last position in such formulas.

33. The same permutation is also extended to integral signs; for, if the integral formula $\int V$ is propounded, while $\int$ denotes the sum of all values taken in the same state, which values correspond to all values of x, it will also be

$$\delta \int V = \int \delta V,$$

which is perspicuous per se, since the increment of the translations of the whole sum is equal to the sum of all elementary increments in the same translation. And from this source the above analysis was derived; for, after the integral formula $\int Zdx$ had been propounded, whose variation in the varied state had to be defined, we assumed

$$\delta \int Zdx = \int \delta(Zdx) = \int \delta Z \cdot dx,$$

since

$$\delta(Zdx) = \delta Zdx + Z\delta dx,$$

but on the other hand $\delta dx = 0$, as $\delta x = 0$. Yes, even if a double integral $\int \int V$ would occur, in like manner it would be

$$\delta \iint V = \int \delta \int V = \iint \delta V.$$

34. Another artifice consists of the transformation of integrals, whenever under the integral sign the signs d and δ are conjugated, that at least in the integration δ is then isolated. So, having propounded the integral formula $\int V\delta dv$, because of $\delta dv = d\delta v$, considering δv as a simple quantity, it will be

$$\int V\delta dv = \int Vd\delta v = V\delta v - \int \delta v dV.$$

And in like manner one will see that

$$\begin{aligned}
&\int Vdd\delta v = Vd\delta V - \delta v dV + \int \delta v ddV,\\
&\int Vd^3\delta v = Vdd\delta v - d\delta v dV + \delta v ddV - \int \delta v d^3 V,\\
&\int Vd^4\delta v = Vd^3\delta v - d^2\delta v dV + d\delta v ddV - \delta v d^3 V + \int \delta d^4 V
\end{aligned}$$

etc.;

for,

$$\int Vdd\delta v = Vd\delta v - \int d\delta v dV,$$

but

$$\int d\delta v dV = \delta v dV - \int \delta v ddV,$$

whence the nature of these transformations is understood.

35. Having mentioned these analytic rules in advance, it will not be difficult to resolve all questions about maxima and minima of this kind, even if in the formula $\int Zdx$ the function Z involves arbitrary integral formulas. The whole task obviously reduces to the definition of the increment $\delta \int Zdx$, which the propounded formula $\int Zdx$ gains, while it is translated from the principal into the varied state; then, this increment put equal to zero will contain the solution of the maximum or minimum. But I will call this increment the *differential variation* of the formula $\int Zdx$, which is to be understood to arise, if each value of y is increased by the infinitely small and arbitrary segments δy. But then it is perspicuous that this variation must be extended throughout all values of x from the limit $x = 0$ to the limit $x = a$, for which complete determination it is to be noted it is to be taken in such a way that it vanishes for $x = 0$. Therefore, hence let us resolve the following problems by applying this method, on which it is to be noted that the letters p, q, r, s etc. involve the ratio of differentials of the two variables x and y in such a way that

$$p = \frac{dy}{dx}, \quad q = \frac{dp}{dx}, \quad r = \frac{dq}{dx}, \quad s = \frac{dr}{dx} \quad \text{etc.}$$